Greatest Common Factor and Least Common Multiple

Intro

In order to understand the concepts of Greatest Common Factor (GCF) and Least Common Multiple (LCM), we need to define two key terms:

**Multiple**: Multiples of a number are the result of multiplying that number by some other whole number.

For example, the table below shows the first 11 multiples of 4:

<table>
<thead>
<tr>
<th>x</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
<th>7</th>
<th>8</th>
<th>9</th>
<th>10</th>
<th>11</th>
</tr>
</thead>
<tbody>
<tr>
<td>4</td>
<td>8</td>
<td>12</td>
<td>16</td>
<td>20</td>
<td>24</td>
<td>28</td>
<td>32</td>
<td>36</td>
<td>40</td>
<td>44</td>
</tr>
</tbody>
</table>

**Factor**: Factors are the numbers you multiply to get another number. For example, the factors of 21 are 3 and 7 because $3(7) = 21$.

A factor of a number will also divide that number evenly, with no remainder. For example, 3 is a factor of 15 because $\frac{15}{3} = exactly \ 5$.

Greatest Common Factor

The Greatest Common Factor (GCF) of two or more numbers is the product of all the factors they share in common.

In order to find the factors they share in common, we use something called prime factorization to find the number’s prime factors.

We can find a number’s prime factors using something called a *factor tree*. The next section will give you an explanation of how to create your own factor trees.
Factor Tree

Prime factorization (or finding primes factors) can be done using a factor tree. Suppose you wanted to find the prime factors of 12. To do so, we will create a factor tree:

```
  12
   /\   \
  /   \  
6     2
  /\   /\ \
 / \ / \ / \
3  2 2
```

- Write your number at the top.
- Next, draw two “branches” and write two factors from the number atop.
- Keep creating branches and “breaking” up the previous numbers until you reach a prime factor. Once you reach a prime factor, the branch ends.

So, in this example, notice that I first “branched” 12 into 2 and 6. Since 2 is a prime factor, my branch stops. However, I can still “branch” 6—into 2 and 3. Since 2 and 3 are prime factors, those branches stop. Once I can not branch any more, I have found my prime factors.

**Therefore, the prime factors of 12 are** $2 \times 2 \times 3$. Notice we write them with multiplication symbols to denote the fact that if I multiply these numbers I will get 12.

**Quick note:** I could have branched 12 into 3 and 4 instead of 2 and 6. Either way you choose, the answer will always be the same. Take a look:

```
  12
   /\   \
  /   \  
6     2
  /\   /\ \
 / \ / \ / \
3  2 2
```

Either way we begin, we will always get the same result.

Let’s try a few others to make sure we understand how to find prime factors.
Example 1

Find the prime factors of 24.

Solution

To find the prime factors of 24, I created a factor tree. Notice I began with the factors 4 and 6, you could have used 2 and 12 or 3 and 8—really up to you.

My factor tree ended with the prime factors of 24:

\[
2 \times 2 \times 2 \times 3
\]

Example 2

Find the prime factors of 62.

Solution

To find the prime factors of 64, I created a factor tree. I started my tree with the factors 2 and 32, however I could have started with 4 and 16 or 8 and 8.

My factor tree ended with the prime factors of 64:

\[
2 \times 2 \times 2 \times 2 \times 2 \times 2
\]
In order to find the GCF of two or more numbers, we will use prime factors.

Suppose we are asked to find the GCF of 10 and 25. To do so, let’s first find the prime factors of both numbers:

\[
\begin{align*}
10: & \quad 2 \times 5 \\
25: & \quad 5 \times 5
\end{align*}
\]

The GCF is the product of the factors they share in common. Notice that both numbers share the factor 5, therefore the **GCF of 10 and 25 is 5**.

Let’s try another example: Find the GCF of 12 and 20. First, find the prime factors of both numbers,

\[
\begin{align*}
12: & \quad 2 \times 2 \times 3 \\
20: & \quad 2 \times 2 \times 5
\end{align*}
\]

Remember, the GCF is the *product* of the factors they share in common. Notice both numbers share the factor 2 twice, therefore the **GCF of 12 and 20 is 2 \times 2 = 4**.

One final example: Find the GCF of 36 and 54.

\[
\begin{align*}
36: & \quad 2 \times 2 \times 3 \times 3 \\
54: & \quad 2 \times 3 \times 3 \times 3
\end{align*}
\]

Notice both numbers share the factor 2 once and the factor 3 twice, therefore **GCF of 36 and 54 is 2 \times 3 \times 3 = 18**.
Least Common Multiple

The Least Common Multiple (LCM) is the *smallest* number that is a multiple of two or more numbers.

For example, suppose we wanted to find the LCM of 3 and 4. To do this, we list the multiples of 3 and 4 until we find the *smallest* number (multiple) both share in common.

<table>
<thead>
<tr>
<th></th>
<th>3</th>
<th>6</th>
<th>9</th>
<th>12</th>
<th>15</th>
<th>16</th>
</tr>
</thead>
<tbody>
<tr>
<td>4</td>
<td>8</td>
<td>12</td>
<td>16</td>
<td>20</td>
<td>24</td>
<td></td>
</tr>
</tbody>
</table>

In this case, **12** is the smallest multiple (LCM) of 3 and 4. Notice both numbers also share 16 as a common multiple, however when finding the LCM we always look for the *smallest* multiple.

Let’s try another example,

Suppose we wanted to find the LCM of 12 and 20. First, write the first few multiples of each number until we find the smallest number both share in common.

<table>
<thead>
<tr>
<th></th>
<th>12</th>
<th>24</th>
<th>36</th>
<th>48</th>
<th>60</th>
<th>72</th>
</tr>
</thead>
<tbody>
<tr>
<td>20</td>
<td>40</td>
<td>60</td>
<td>80</td>
<td>100</td>
<td>120</td>
<td></td>
</tr>
</tbody>
</table>

**The LCM of 12 and 20 is 60.**

Writing the multiples of each number and finding the smallest one they share in common is pretty straightforward. However, if we begin working with large numbers it can take some time finding each multiple. Another method for finding the LCM uses each number’s prime factors.
Suppose we wanted to find the LCM of 30 and 45. Instead of creating a table which lists each number’s multiples; let’s write the prime factors of each number (*we can find these using our factor tree*).

<table>
<thead>
<tr>
<th></th>
<th>2</th>
<th>3</th>
<th>5</th>
</tr>
</thead>
<tbody>
<tr>
<td>30</td>
<td>2</td>
<td>3</td>
<td>5</td>
</tr>
<tr>
<td>45</td>
<td>3</td>
<td>3</td>
<td>5</td>
</tr>
</tbody>
</table>

Next, multiply each factor the greatest number of times it occurs for either number. That is:

\[
\begin{align*}
2 & \text{ appears once} \\
3 & \text{ appears twice} \\
5 & \text{ appears once}
\end{align*} \]

\[
2 \times 3 \times 3 \times 5 = 90
\]

Now, you might be thinking, “Hey wait a minute, I can see how 2 appears once, but 3 appears 3 times! What gives?” Well, remember, we only write the greatest number of times a factor appears either for 30 or 45. Since 3 appears twice for the number 30 and only once for the number 45, we use 3 twice (*remember, we want the greatest number of times it appears for either 30 or 45, not both combined*).

Here’s another example using this method: Find the LCM of 12 and 80.

First, find the prime factors of both 12 and 80 (*we can use our factor tree for this*).

<table>
<thead>
<tr>
<th></th>
<th>2</th>
<th>2</th>
<th>3</th>
</tr>
</thead>
<tbody>
<tr>
<td>12</td>
<td>2</td>
<td>2</td>
<td>3</td>
</tr>
<tr>
<td>80</td>
<td>2</td>
<td>2</td>
<td>2</td>
</tr>
</tbody>
</table>

Next, multiply each factor the greatest number of times it appears for either number:

\[
\begin{align*}
2 & \text{ appears 4 times} \\
3 & \text{ appears once} \\
5 & \text{ appears once}
\end{align*} \]

\[
2 \times 2 \times 2 \times 2 \times 3 \times 5 = 240
\]

**Think Back…**

A prime number is a number with exactly two factors, 1 and itself.

For example, 5 is a prime number because its factors are 1 and 5 (itself).
One final example using this method: Find the LCM of 36 and 60.

First, find the prime factors of both 36 and 60 (use a factor tree).

<table>
<thead>
<tr>
<th></th>
<th>2</th>
<th>2</th>
<th>3</th>
<th>3</th>
</tr>
</thead>
<tbody>
<tr>
<td>36</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>60</td>
<td></td>
<td></td>
<td></td>
<td>5</td>
</tr>
</tbody>
</table>

Next, multiply each factor the greatest number of times it appears for either number:

2 appears twice (same number of times for both 36 and 60)
3 appears twice
5 appears once

\[ 2 \times 2 \times 3 \times 3 \times 5 = 180 \]

You can use whichever method is easier for you. The first method is straightforward because you just write down all of the multiples for either number and find the smallest one in common. However, when working with large numbers this method can become tedious. The second method involves a couple of more steps, however it makes working with larger values much easier.